熱保存則から

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \frac{1}{\rho c_p}\frac{\partial Q}{\partial z} + \frac{\partial}{\partial z}\left(\kappa\frac{\partial T}{\partial z}\right)$$

鉛直方向に積分し，連続方程式 $\nabla\cdot\mathbf{u} = 0$ を用いると，

$$\int_{-h}^{0}\frac{\partial T}{\partial t}dz + \int_{-h}^{0}\nabla\cdot(\mathbf{u}T)dz = \frac{1}{\rho c_p}\int_{Q_{pen}}^{Q_0}\partial Q + \int_{\kappa\frac{\partial T}{\partial z}|_{z=-h}}^{0}\partial\left(\kappa\frac{\partial T}{\partial z}\right) \tag{1}$$

ここで，2 変数の部分積分は

$$\frac{\partial}{\partial x}\int_{a}^{g(x,t)} f(x,t)dt = \int_{a}^{g(x,t)}\frac{\partial f}{\partial x}dt + f(x,g)\frac{\partial g}{\partial x}$$

となる．

［証明］

$$\begin{aligned}
\frac{\partial}{\partial x}\int_{a}^{g(x,t)} f(x,t)dt &= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\int_{a}^{g(x+\Delta x,t)} f(x+\Delta x,t)dt - \int_{a}^{g(x,t)} f(x,t)dt\right)\\
&= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\int_{z}^{g(x+\Delta x,t)}(f(x+\Delta x,t) - f(x,t))dt\right)\\
&+ \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\int_{a}^{g(x+\Delta x,t)} f(x,t)dt - \int_{a}^{g(x,t)} f(x,t)dt\right)\\
(1st\ term\ on\ rhs) &= \int_{a}^{g(x,t)}\frac{\partial f}{\partial x}dt\\
(2nd\ term\ on\ rhs) &= \lim_{\Delta\to 0}\frac{\int_{g(x,t)}^{g(x+\Delta x,t)} f(x,t)dt}{g(x+\Delta x,t) - g(x,t)}\frac{g(x+\Delta x,t) - g(x,t)}{(x+\Delta x) - x}\\
&= f(x,g(x,t))\frac{\partial g}{\partial x}
\end{aligned}$$

故に，

$$\frac{\partial}{\partial x}\int_{a}^{g(x,t)} f(x,t)dt = \int_{a}^{g(x,t)}\frac{\partial f}{\partial x}dt + f(x,g)\frac{\partial g}{\partial x}$$

これを用いて，

$$\int_{-h}^{0}\frac{\partial}{\partial x}(uT)dz=\frac{\partial}{\partial x}\int_{-h}^{0}uTdz-u_{-h}T_{-h}\frac{\partial h}{\partial x} \tag{2}$$

$$\int_{-h}^{0}\frac{\partial}{\partial y}(vT)dz=\frac{\partial}{\partial y}\int_{-h}^{0}vTdz-v_{-h}T_{-h}\frac{\partial h}{\partial y} \tag{3}$$

$$\int_{-h}^{0}\frac{\partial}{\partial z}(wT)dz=-w_{-h}T_{-h} \tag{4}$$

ここで，ある変数 x を混合層内で鉛直平均したものを

$$x_a=\frac{1}{h}\int_{-h}^{0}xdz$$

と書く．よって，

$$\int_{-h}^{0}\frac{\partial T}{\partial t}dz=h\frac{\partial T_a}{\partial t}+T_a\frac{\partial h}{\partial t}-T_{-h}\frac{\partial h}{\partial t} \tag{5}$$

ここで，(2)，(3) 式の右辺第 2 項を u_a や T_a で書き表す．

$$\frac{\partial}{\partial x}\left(u_a\int_{-h}^{0}Tdz\right)=\int_{-h}^{0}udz\frac{\partial T_a}{\partial x}+T_a\frac{\partial}{\partial x}\int_{-h}^{0}udz \tag{6}$$

$$\frac{\partial}{\partial y}\left(v_a\int_{-h}^{0}Tdz\right)=\int_{-h}^{0}vdz\frac{\partial T_a}{\partial y}+T_a\frac{\partial}{\partial y}\int_{-h}^{0}vdz \tag{7}$$

連続方程式から

$$\int_{-h}^{0}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)dz=0$$

$$w_{-h}=\frac{\partial}{\partial x}\int_{-h}^{0}udz-u_{-h}\frac{\partial h}{\partial x}+\frac{\partial}{\partial y}\int_{-h}^{0}vdz-v_{-h}\frac{\partial h}{\partial y} \tag{8}$$

(6),(7),(8)$\times T_a$ から

$$\begin{aligned}\int_{-h}^{0}udz\frac{\partial T_a}{\partial x}-\frac{\partial}{\partial x}\left(T_a\int_{-h}^{0}udz\right)+\int_{-h}^{0}vdz\frac{\partial T_a}{\partial y}-\frac{\partial}{\partial y}\left(T_a\int_{-h}^{0}vdz\right)\\+T_a\left(u_{-h}\frac{\partial h}{\partial x}+v_{-h}\frac{\partial h}{\partial y}+w_{-h}\right)=0\end{aligned} \tag{9}$$

また，

$$\frac{\partial}{\partial x}\left(u_a T_a \int_{-h}^{0} dz\right) = \frac{\partial}{\partial x}\left(u_a \int_{-h}^{0} T dz\right) \tag{10}$$

$$\frac{\partial}{\partial y}\left(v_a T_a \int_{-h}^{0} dz\right) = \frac{\partial}{\partial y}\left(v_a \int_{-h}^{0} T dz\right) \tag{11}$$

(1) に (2)-(5) を代入，(9)-(11) を足すと

$$\begin{aligned}\frac{\partial T_a}{\partial t} + \mathbf{u_a}\cdot\nabla T_a + \nabla\cdot\int_{-h}^{0}\frac{\mathbf{u}'T'}{h}dz + \frac{T_a - T_{-h}}{h}\left(\frac{\partial h}{\partial t} + \mathbf{u_{-h}}\cdot\nabla h + w_{-h}\right) \\ = \frac{Q_0 - Q_{pen}}{\rho c_p h} - \frac{\kappa}{h}\frac{\partial T}{\partial z}|_{z=-h}\end{aligned} \tag{12}$$

ここで，$\mathbf{u}', T'$ は

$$\mathbf{u} = \mathbf{u_a} + \mathbf{u}'$$
$$T = T_a + T'$$

混合層水温収支解析を行う際，シア流，鉛直拡散などを残差に含めて，

$$\frac{\partial T_a}{\partial t} = \frac{Q_0 - Q_{pen}}{\rho c_p h} - \mathbf{u_a}\cdot\nabla T_a - \left(w_{-h} + \frac{\partial h}{\partial t}\right)\frac{T_a - T_{-h}}{h} + res \tag{13}$$

として扱う．
蓄熱量収支解析の際，h が定数 H となるので，

$$\begin{aligned}\frac{\partial}{\partial t}\int_{-H}^{0}\rho c_p T dz + \mathbf{u_a}\cdot\int_{-H}^{0}\rho c_p T dz + \nabla\cdot\int_{-H}^{0}\rho c_p \mathbf{u}'T' dz + \rho c_p w_{-H}(T_a - T_{-H}) \\ = Q_0 - Q_{pen} - \rho c_p \kappa\frac{\partial T}{\partial z}|_{z=-H}\end{aligned} \tag{14}$$

H を STMW の中心と設定するので，鉛直拡散，Q_{pen} は小さいと考えられる．
ここで，水平移流に関する項を残差として，見積もるとして，

$$\frac{\partial}{\partial t}\int_{-H}^{0}\rho c_p T dz = Q_0 - w_{-H}\rho c_p (T_a - T_{-H}) + res \tag{15}$$